

Vector bundles and the icosahedron

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Dedicated to S. Ramanan on the occasion of his 70th birthday

1 Introduction

Describing the equation of a plane curve C as the determinant of a matrix of linear forms is a classical problem. A related issue is expressing C as the Pfaffian of a skew-symmetric matrix of forms. When the curve is defined by a harmonic homogeneous polynomial (a solution to Laplace's equation), then the action of the Lie algebra of the orthogonal group $SO(3, \mathbf{C})$ provides a natural such expression, and with it, a natural semi-stable rank two vector bundle E on C . This paper concerns itself with these bundles, which have trivial determinant and $H^0(C, E) = 0$.

For a cubic curve, Atiyah's classification shows that they fall into three types: a sum of line bundles $L \oplus L^*$ with L^2 nontrivial, a non-trivial extension of a line bundle of order two by itself, and a trivial extension. We shall show that the second and third cases are described in the space of harmonic cubics by the vanishing of a certain invariant polynomial of degree six.

Our route to this result uses the Mukai-Umemura threefold and we link up with the classical geometry of the Clebsch diagonal cubic surface, certain distinguished rational curves on it, and the action of the symmetric group S_5 .

The original motivation for this paper comes from the author's interest in explicit solutions to Painlevé equations, as in [10] and [11]. These were connected to the study of certain threefolds with an open orbit of $SO(3, \mathbf{C})$ and finite stabilizer. The Mukai-Umemura manifold has such an action where the stabilizer is the icosahedral group. Since the first paper, where the stabilizer is the dihedral group, was written for the 60th birthdays of Narasimhan and Seshadri, it seems appropriate, sixteen years

on, to discuss the icosahedron for Ramanan.

2 Representations of $SO(3, \mathbf{C})$

Let V be a finite-dimensional irreducible representation space of the group $SO(3, \mathbf{C})$. These occur in each odd dimension $(2d+1)$ and have an invariant inner product (u, v) . Restricted to the circle subgroup $SO(2, \mathbf{C}) \subset SO(3, \mathbf{C})$, the weights are $-d \leq n \leq d$; in particular the weight zero occurs with multiplicity one so there is a unique invariant element. The usual realization of V is as S^{2d} – the space of homogeneous polynomials in the complex variables z_1, z_2 of degree $2d$ under the action of $SL(2, \mathbf{C})/\pm 1 \cong SO(3, \mathbf{C})$.

The Lie algebra $\mathfrak{so}(3)$ acts on V as skew-adjoint transformations and so each $x \in \mathfrak{so}(3)$ defines a skew form

$$\omega_x(u, v) = (x \cdot u, v).$$

Now fix $v \in V$ and restrict ω_x to the $2d$ -dimensional orthogonal complement W of v . The Pfaffian $\omega_x \wedge \omega_x \dots \wedge \omega_x \in \Lambda^{2d} W^* \cong \mathbf{C}$ defines a homogeneous polynomial $f(x)$ of degree d .

The fact that we have a natural map from vectors in V to polynomials is no surprise if we recall that another realization of the $(2d+1)$ -dimensional irreducible representation of $SO(3, \mathbf{C})$ is as spherical harmonics – homogeneous polynomials $f(x_1, x_2, x_3)$ of degree d which satisfy Laplace's equation. However, v gives somewhat more than just a polynomial.

The skew form ω_x defines a map $\omega_x : W \rightarrow W^*$, which is linear in x . Hence if \mathbf{P}^2 is the projective space of the vector space $\mathfrak{so}(3)$, we have a sequence of sheaves

$$0 \rightarrow \mathcal{O}_{\mathbf{P}^2}(W(-2)) \xrightarrow{\omega} \mathcal{O}_{\mathbf{P}^2}(W^*(-1)) \rightarrow E \rightarrow 0 \quad (1)$$

where E is a rank 2 vector bundle supported on the curve $C \subset \mathbf{P}^2$ defined by $f(x) = 0$. The vector bundle satisfies $\Lambda^2 E \cong K_C$, the canonical bundle, and from the exact cohomology sequence, $H^0(C, E) = 0$.

These facts about curves defined by Pfaffians (and much more besides) can be found in [3]. In particular, the bundle is always semi-stable. To see this, note that if $L \subset E$ is a subbundle, the inclusion defines a section i of L^*E and if $\deg L \geq g$ then by Riemann-Roch there is a non-zero section s of L . But then si is a nonzero section of E which is a contradiction. Hence we must have

$$\deg L \leq g - 1 = \frac{1}{2} \deg K_C = \frac{1}{2} \deg E.$$

This natural process thus generates a curve and a rank 2 semi-stable bundle, and the question we ask is “What is this bundle?”

When $d = 3$, the curve C is a plane cubic and then we know from Atiyah’s classification of bundles on an elliptic curve [1] that, when C is smooth, there are only three possibilities. Since $K_C \cong \mathcal{O}$, $\Lambda^2 E$ is trivial and $H^0(C, E) = 0$ and we have the cases:

- $E = L \oplus L^*$ where $\deg L = 0$ and $L \neq \mathcal{O}$ (the generic case)
- E is a non-trivial extension $0 \rightarrow L \rightarrow E \rightarrow L \rightarrow 0$ where $L^2 \cong \mathcal{O}$ and $L \neq \mathcal{O}$
- $E = L \oplus L$ where $L^2 \cong \mathcal{O}$ and $L \neq \mathcal{O}$.

These three types define a stratification of the six-dimensional projective space $\mathbf{P}(V)$ where V is the seven-dimensional space of harmonic cubic polynomials f – or at least that part of it for which the curve $f(x) = 0$ is nonsingular. To investigate this further we have to see a link with another piece of geometry.

3 Isotropic spaces

Let $d = 3$, and as above let $f \in V$ define a smooth cubic curve $C \subset \mathbf{P}^2$ and $W \subset V$ the six-dimensional orthogonal complement of f in W . Since $\Lambda^2 E \cong \mathcal{O}$, we have $E^* \cong E$ and then the kernel of $\omega : W \rightarrow W^*(1)$ on C can be identified with $E(-1)$.

Suppose $U \subset W$ is a three-dimensional subspace, *isotropic* with respect to the skew forms ω_x for all x . Then, for each x , $\omega_x : W \rightarrow W^*$ maps U to its annihilator U° and so defines a homomorphism of sheaves on \mathbf{P}^2 , $\omega|_U : U \rightarrow U^\circ(1)$. Moreover the determinant of this map is the Pfaffian of ω_x . The kernel of $\omega|_U$ on C is then a line bundle $L(-1) \subset U$. Since this is the restriction of ω to U , L is a subbundle of E . From the treatment of determinantal loci in [3] we have $\deg L = 0$ and $H^0(C, L) = 0$ so $L \neq \mathcal{O}$.

Conversely, on C , suppose we have a line bundle $L \subset E$ with $\deg L = 0$ and $L \neq \mathcal{O}$. From the exact cohomology sequence of (1) we have $W^* \cong H^0(C, E(1))$. If $\alpha \in W^*$ annihilates $L(-1)$ then it defines a section of $(E/L)(1) = L^*(1)$ which is of degree 3 and so has a three-dimensional space of sections on C . It follows that $L(-1) \subset W$ sweeps out a three-dimensional subspace U .

Now $\omega|_U$ is a section of $\Lambda^2 U^*(1)$ on C and we have $\Lambda^2 U^* \cong U \otimes \Lambda^3 U^*$ so this defines a section w of $U(1)$ which tautologically is annihilated by ω . Thus if $w \neq 0$, then

$\ker \omega|_U \cong \mathcal{O}(-1)$. But $\ker \omega|_U \cong L(-1)$ and $L \neq \mathcal{O}$, so we deduce that $w = 0$ and U is isotropic for each ω_x .

If we look again at the trichotomy of Atiyah's classification, we see that the three cases are equivalent to

- E has two rank one subbundles
- E has one rank one subbundle
- E has infinitely many rank one subbundles

and by what we have just seen, this condition translates into an equivalent statement about the three-dimensional isotropic subspaces of W :

- W has two isotropic subspaces
- W has one isotropic subspace
- W has infinitely many isotropic subspaces.

This provides an extension of the criterion to *any* $f \in V$, and not just those which define smooth cubic curves. (In fact, Atiyah's classification has been extended in [5] to a class of singular elliptic curves; moreover the wild case of cuspidal cubics does not occur when f is harmonic).

4 The Mukai-Umemura threefold

The study of three-dimensional isotropic subspaces of V is best approached via a special Fano threefold introduced by Mukai and Umemura [15]. We consider the Grassmannian $G(3, V)$ of three-dimensional subspaces of the seven-dimensional representation space V and its universal rank 3 bundle \mathcal{E} . For $x \in \mathfrak{so}(3)$ we have the skew form ω_x on V , and so a section of the rank 9 vector bundle over $G(3, V)$:

$$\Lambda^2 \mathcal{E}^* \otimes \mathfrak{so}(3).$$

Now $\dim G(3, V) = 3 \times (7 - 3) = 12$ and then the zero set of the section is a smooth $12 - 9 = 3$ -dimensional manifold known as the *Mukai-Umemura threefold* Z . By construction it parametrizes subspaces U isotropic for all ω_x and thereby has a natural action of $SO(3, \mathbf{C})$.

The threefold Z has the same additive integral cohomology as \mathbf{P}^3 and $H^2(Z, \mathbf{Z})$ is generated by $x = c_1$. However $x^2 \in H^4(Z, \mathbf{Z})$ is 22 times a generator y . Since $c_1 > 0$ the Todd genus is 1 so $c_1 c_2 = 24$ and we have Chern classes

$$c_1 = x \quad c_2 = 24y \quad c_3 = 4xy. \quad (2)$$

Given $f \in V$, the inner product $(f, -)$ defines an element of V^* and, restricting to the universal bundle $\mathcal{E} \subset V$, we get a section of \mathcal{E}^* . Over Z , this is a rank three bundle and the section vanishes at the points which correspond to isotropic three-dimensional subspaces orthogonal to f .

Proposition 1 *On the Mukai-Umemura threefold, $c_3(\mathcal{E}^*) = 2$.*

Proof: The tangent bundle of the Grassmannian is $\text{Hom}(\mathcal{E}, V/\mathcal{E})$ and Z is the non-degenerate zero set of a section of $\Lambda^2 \mathcal{E}^* \otimes \mathbf{C}^3$. Thus, as C^∞ bundles

$$TZ \oplus (\Lambda^2 \mathcal{E}^* \otimes \mathbf{C}^3) \cong \text{Hom}(\mathcal{E}, V/\mathcal{E}).$$

Applying the Chern character we find

$$c_1(\mathcal{E}^*) = c_1 \quad c_2(\mathcal{E}^*) = c_1^2 - \frac{1}{2}c_2 \quad c_3(\mathcal{E}^*) = \frac{1}{10}(c_3 + 4c_1^3 - 3c_1 c_2).$$

From (2) we obtain $10c_3(\mathcal{E}^*) = 4 + 88 - 72 = 20$ and hence the result. \square

A section with nondegenerate zero set will thus vanish at two points. We can now see the trichotomy in terms of the section s of \mathcal{E}^* :

- s vanishes at two points
- s vanishes at one point
- s vanishes on a subvariety of positive dimension.

5 The icosahedron

The ten-dimensional space of all homogenous cubics in x has an $SO(3, \mathbf{C})$ -invariant product which can be normalized so that $(f(x), (x, a)^3) = f(a)$. For fixed $a \in \mathbf{C}^3$, the polynomial $(x, a)^3$ is not harmonic but its orthogonal projection onto V is

$$f_a(x) = (x, a)^3 - \frac{3}{5}(a, a)(x, a)(x, x).$$

This is invariant by the group of rotations fixing a . For any cubic f we still have the inner product property $(f, f_a) = f(a)$.

The action of $u \in \mathfrak{so}(3)$ on f_a is

$$(u \cdot f_a)(x) = 3(x, [u, a])[(x, a)^2 - \frac{1}{5}(a, a)(x, x)].$$

Take a regular icosahedron with vertices at $\pm a_1, \pm a_2, \dots, \pm a_6$. The angle θ between any two axes joining opposite vertices satisfies $\cos \theta = \pm 1/\sqrt{5}$. Now consider

$$(u \cdot f_{a_i}, f_{a_j}) = 3(a_j, [u, a_i])[(a_j, a_i)^2 - \frac{1}{5}(a_i, a_i)(a_j, a_j)].$$

If $i = j$ then $(a_j, [u, a_i]) = 0$ and when $i \neq j$

$$(a_j, a_i)^2 - \frac{1}{5}(a_i, a_i)(a_j, a_j) = (\cos^2 \theta - \frac{1}{5})(a_i, a_i)(a_j, a_j) = 0$$

so f_{a_1}, \dots, f_{a_6} span an isotropic subspace U , invariant under the icosahedral group G . Its orthogonal complement in V consists of those f such that

$$0 = (f, f_{a_i}) = f(a_i) = 0.$$

There are five objects which are permuted by G which realize the well-known isomorphism $G \cong A_5$. These are the five sets of three orthogonal planes in which all 12 vertices lie. Let e_1, e_2, e_3 be the three unit normals to such a set then the cubic $(e_1, x)(e_2, x)(e_3, x)$ vanishes at a_i as do all its transforms by G . These span the four-dimensional permutation representation **4** of A_5 . Moreover, since $(e_1, x)(e_2, x)(e_3, x)$ satisfies Laplace's equation, this is the orthogonal complement of U in V , hence U has dimension 3. Every icosahedron therefore defines a point in the Mukai-Umemura manifold and the three-dimensional orbit $SO(3, \mathbf{C})/G$ is a dense open set in Z .

Generically, a harmonic cubic f defines a section s of \mathcal{E}^* which vanishes at two points in this open set, which means that the corresponding plane cubic C contains the twelve points in \mathbf{P}^2 defined by the axes of two icosahedra. Put another way, if a generic cubic curve contains one such “icosahedral set” $\{[a_1], [a_2], \dots, [a_6]\} \subset \mathbf{P}^2$ then it contains another.

6 Degenerate icosahedra

6.1 The two types

We need to understand also the divisor of “degenerate icosahedral sets” which forms the complement of the open orbit of $SO(3, \mathbf{C})$ in Z . These correspond to isotropic

subspaces of two types, constituting two orbits, of dimension 2 and 1 respectively.

A representative of the first type is the space with basis

$$(b, x)^3 - \frac{3}{5}(b, x)(b, b)(x, x), \quad (a, x)^2(b, x), \quad (a, x)^3$$

where $(a, a) = 0$, $(a, b) = 0$ and $(b, b) \neq 0$. This subspace is invariant by the rotations about the axis b and is spanned by the weight spaces $\{0, 2, 3\}$ for that action. Geometrically, the isotropic subspace is defined by $[b] \in \mathbf{P}^2$ (a point not on the null conic Q defined by $(x, x) = 0$) together with the point of contact $[a]$ of a tangent to Q through $[b]$. The choice of tangent is a double covering of $\mathbf{P}^2 \setminus Q$, which is abstractly an affine quadric.

The icosahedral set here describes five of the vectors $\{a_1, a_2, \dots, a_6\}$ coalescing into a single vector a . Under this degeneration the relation $(a_j, a_i)^2 - \frac{1}{5}(a_i, a_i)(a_j, a_j) = 0$ implies that a is null and the sixth vector b satisfies $(b, a) = 0$.

The second type is a subspace with basis

$$(a, x)((b, x)^2 - \frac{1}{5}(b, b)(x, x)), \quad (a, x)^2(b, x), \quad (a, x)^3$$

where $(a, a) = 0$ and $(a, b) = 0$. This is geometrically defined by the point $[a] \in Q$ and is invariant by the Borel subgroup which fixes $[a]$. Decomposing with respect to a semisimple element this is the span of weight spaces $\{1, 2, 3\}$. Here all six vectors $\{a_1, a_2, \dots, a_6\}$ coalesce into a single null vector a .

The union of these two orbits forms an anticanonical divisor in Z , for if X_1, X_2, X_3 are the vector fields on Z generated by a basis of $\mathfrak{so}(3)$, then $X_1 \wedge X_2 \wedge X_3$ vanishes on the lower-dimensional orbits and this is a section of $K_Z^* = \Lambda^3 T^*Z$. Since $c_1(Z)$ is a generator of $H^2(Z, \mathbf{Z})$ it must vanish with multiplicity 1. Note that this anticanonical divisor D cannot be smooth (for then it would be a K3 surface or a torus). It is instead a singular image of a map $\alpha : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow Z$ (in fact, a transverse to the diagonal in $\mathbf{P}^1 \times \mathbf{P}^1$ has a cusp singularity $y^2 = x^3$ (see [7])).

To define α recall that we have identified the type 1 orbit as an affine quadric, the complement of the diagonal in $\mathbf{P}^1 \times \mathbf{P}^1$. This is an ordered pair of points $[a], [a']$ on the conic $Q \cong \mathbf{P}^1$, where the two tangents meet at $[b] \in \mathbf{P}^2$. To extend the map α to the diagonal put $c = a + tb$, with $(c, a) = 0$ and let $t \rightarrow 0$. Then

$$(c, x)^3 - \frac{3}{5}(c, x)(c, c)(x, x) = (a, x)^3 + 3t(a, x)^2(b, x) + 3t^2[(a, x)((b, x)^2 - \frac{1}{5}(b, b)(x, x))].$$

and the three leading coefficients span the type 2 isotropic subspace.

Consider a degenerate icosahedral set of type 1, spanned by the three cubic polynomials $(b, x)^3 - 3(b, x)(b, b)(x, x)/5$, $(a, x)^2(b, x)$ and $(a, x)^3$. Then f is orthogonal to this if the curve C given by $f(x) = 0$ intersects the conic Q tangentially at $[a]$ and passes through $[b]$. For type 2, the intersection multiplicity of C with Q at $[a]$ must be at least 3.

6.2 The universal bundle

We should also consider the universal bundle \mathcal{E} on the divisor D , or rather its pullback $\alpha^*\mathcal{E}$ on $\mathbf{P}^1 \times \mathbf{P}^1$. Note that both types of degenerate isotropic subspace contain $(a, x)^3$ and $(a, x)^2(b, x)$ where $(b, a) = 0$, or equivalently the two-dimensional subspace given by $(a, x)^2(c, x)$ for all c orthogonal to a .

We use the two-fold covering map $\pi : \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow \mathbf{P}^2$, the quotient by the involution interchanging the factors. Each factor is isomorphic to the diagonal which maps to the conic Q . As usual, geometrically $\pi([a], [a'])$ is the point $[b]$ of intersection of the tangents at $[a], [a'] \in Q$.

The two-dimensional vector space $a^\perp \subset \mathbf{C}^3$ for $[a] \in Q$ defines a vector bundle A over Q for which the projective bundle $\mathbf{P}(A)$ is trivial. Indeed, it is the bundle of tangent lines to the conic and so under the map π can be identified with the family $\mathbf{P}^1 \times \{x\}$, $x \in \mathbf{P}^1$, which is a trivial bundle over the diagonal. On the other hand since $(a, a) = 0$

$$a^\perp \cong (\mathbf{C}^3 / \mathcal{O}_{\mathbf{P}^2}(-1))^*$$

and so $\Lambda^2 A \cong \mathcal{O}_{\mathbf{P}^2}(-1) = \mathcal{O}(-2)$, identifying Q with \mathbf{P}^1 . Hence, since $\mathbf{P}(A)$ is trivial,

$$A \cong \mathcal{O}(-1) \otimes \mathbf{C}^2.$$

Multiplying by the factor $(a, x)^2$ which is quadratic in a , it follows that the subbundle in $\alpha^*\mathcal{E}$ of cubics of the form $(a, x)^2(c, x)$ with $(c, a) = 0$ is isomorphic to $\mathcal{O}(-5, 0) \otimes \mathbf{C}^2$.

The cubic $(b, x)^3 - 3(b, x)(b, b)(x, x)/5$, together with the subspace A , spans a type 1 degenerate subspace for $(b, b) \neq 0$ and when $(b, b) = 0$ it lies in A . This term is homogeneous of degree 3 in b , and so defines a homomorphism from $\pi^*\mathcal{O}_{\mathbf{P}^2}(-3) = \mathcal{O}(-3, -3)$ to $\alpha^*\mathcal{E}$. It projects to a section of the line bundle $\alpha^*\mathcal{E}/A$ which vanishes on the diagonal $\Delta \subset \mathbf{P}^1 \times \mathbf{P}^1$. Hence $\alpha^*\mathcal{E}/A$ is of the form $\mathcal{O}(k, k)$ and

$$\Lambda^3 \alpha^*\mathcal{E} \cong \mathcal{O}(k - 10, k).$$

But for Z we have the Chern number $c_1^3 = 22$ and since D is an anticanonical divisor, $c_1^2[D] = 22$. However $c_1(\mathcal{E}) = -c_1(Z)$ and so

$$22 = 2k(k - 10)$$

and $k = -1$. Thus $\alpha^*\mathcal{E}$ is an extension

$$0 \rightarrow \mathcal{O}(-5, 0) \otimes \mathbf{C}^2 \rightarrow \alpha^*\mathcal{E} \rightarrow \mathcal{O}(-1, -1) \rightarrow 0.$$

This extension is classified by an element of $H^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(-4, 1)) \otimes \mathbf{C}^2$. It is also by definition $SO(3, \mathbf{C})$ -invariant by the diagonal action on $\mathbf{P}^1 \times \mathbf{P}^1$. But as representation spaces

$$H^1(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(-4, 1)) \otimes \mathbf{C}^2 \cong H^1(\mathbf{P}^1, \mathcal{O}(-4)) \otimes H^0(\mathbf{P}^1, \mathcal{O}(1)) \otimes \mathbf{C}^2 \cong S^2 \otimes S \otimes S$$

and $\alpha^*\mathcal{E}$ is defined by a non-zero vector in the unique invariant one-dimensional subspace.

The sections of $\alpha^*\mathcal{E}^*$ fit into the exact sequence

$$0 \rightarrow H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(1, 1)) \rightarrow H^0(\mathbf{P}^1 \times \mathbf{P}^1, \alpha^*\mathcal{E}^*) \rightarrow H^0(\mathbf{P}^1, \mathcal{O}(5)) \otimes \mathbf{C}^2 \rightarrow 0.$$

As a representation space we then have

$$H^0(\mathbf{P}^1 \times \mathbf{P}^1, \alpha^*\mathcal{E}^*) \cong S \otimes S + S \otimes S^5$$

which contains with multiplicity one the seven-dimensional representation S^6 as a subspace of $S \otimes S^5$. These sections are the restriction to D of the sections $(f, -)$ on Z that we have considered earlier. There is one important consequence of this:

Proposition 2 *A harmonic cubic f is orthogonal to a degenerate isotropic subspace if and only if $\Delta(f) = 0$ for a certain $SO(3, \mathbf{C})$ -invariant polynomial Δ of degree 10.*

Proof: We want to know when a section s of \mathcal{E}^* defined by $(f, -)$ vanishes on D . If it does vanish somewhere then the map to $H^0(\mathbf{P}^1, \mathcal{O}(5)) \otimes \mathbf{C}^2$ gives two sections of $\mathcal{O}(5)$ with a common zero. These arise from $S^6 \subset S^5 \otimes S$ so the condition is that we have a homogeneous polynomial $p(z_1, z_2)$ of degree 6 such that the two partial derivatives $\partial p / \partial z_1, \partial p / \partial z_2$, homogeneous of degree 5, have a common zero. The vanishing of the resultant is the condition. This is a degree 10 polynomial Δ in the coefficients of f , the *discriminant*. Its vanishing implies that there is a point $[a] = [z_1, z_2] \in \mathbf{P}^1$ where the section of $\mathcal{O}(6)$ has a double zero. Then the section of $\mathcal{O}(5, 0) \otimes \mathbf{C}^2$ vanishes on $\{[a]\} \times \mathbf{P}^1 \subset \mathbf{P}^1 \times \mathbf{P}^1$.

The dual of the inclusion $\mathcal{O}(-3, -3) \subset \alpha^*\mathcal{E}$ gives a homomorphism

$$\alpha^*\mathcal{E}^* \rightarrow \mathcal{O}(3, 3)$$

which determines the third component of the section s . It maps $H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{E}^*)$ to

$$H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(3, 3)) \cong S^3 \otimes S^3$$

and our seven-dimensional representation space S^6 maps into the symmetric elements. Such a section is then the pull-back of a section of $\mathcal{O}_{\mathbf{P}^2}(3)$ - the cubic f . In particular, its divisor passes through $([a], [a])$ where the other two components of s vanish.

We see then that the section s vanishes at a point of D if the discriminant vanishes. Geometrically this implies that the cubic curve C meets the null conic tangentially at $[a]$.

□

7 The Clebsch cubic surface

The standard permutation representation of A_5 is the four-dimensional subspace $\mathbf{4} \subset \mathbf{C}^5$ defined by $y_1 + y_2 + \dots + y_5 = 0$. We shall investigate next the geometry of the corresponding three-dimensional projective space \mathbf{P}^3 .

The ring of invariants for A_5 on $\mathbf{4}$ is generated by the elementary symmetric functions $\sigma_2, \sigma_3, \sigma_4, \sigma_5$ in (y_1, y_2, \dots, y_5) and the degree 10 invariant

$$\prod_{i < j} (y_i - y_j).$$

The invariant $y_1^2 + y_2^2 + \dots + y_5^2$ defines a nonsingular quadratic form on $\mathbf{4}$ and then orthogonality of subspaces defines polarity in \mathbf{P}^3 : each point has a polar plane, and each line a polar line. There is a *unique* invariant cubic, which we can write as either $\sigma_3 = 0$ or more usually as

$$y_1^3 + y_2^3 + \dots + y_5^3 = 0.$$

This equation defines the *Clebsch cubic surface* S . It is nonsingular and is invariant under the action of the full symmetric group S_5 .

Consider now six points $[a_1], [a_2], \dots, [a_6] \in \mathbf{P}^2$ forming an icosahedral set. No three are collinear and no conic passes through them all, for then it would be invariant by the icosahedral group and so given by the null conic $(x, x) = 0$; but the vertices a_i are not null. It follows that blowing up the six points gives a non-singular cubic surface in \mathbf{P}^3 . More precisely, the embedding is given by the plane cubic curves that pass through the six points, which is the representation $\mathbf{4}$ of A_5 . Hence, by uniqueness, the invariant cubic surface must be the Clebsch surface S .

The blown up points form six disjoint lines in S , half of a double-six configuration. The other six are given by the proper transforms of conics passing through five of the six points. These we have encountered already:

$$(a_i, x)^2 - \frac{1}{5}(a_i, a_i)(x, x) = 0.$$

Blowing these down gives another map from S to a projective plane $\tilde{\mathbf{P}}^2$. Moreover since the icosahedral group permutes these lines, it is the plane of a three-dimensional representation. These two planes are the projective spaces of the two inequivalent three-dimensional representations $\mathbf{3}$ and $\tilde{\mathbf{3}}$ of A_5 . (These can be viewed as the self-dual and anti-self-dual two-forms on $\mathbf{4}$).

Since the null conic $Q \subset \mathbf{P}^2$ does not meet the points $[a_i]$ to be blown up, it lifts to a rational curve $R \subset S$ with self-intersection number 4, and hence of degree 6 in \mathbf{P}^3 . The null conic in the second projective plane similarly lifts to a degree 6 curve $\tilde{R} \subset S$. Each of these is individually invariant by the group A_5 and interchanged by S_5 .

The rational curves R and \tilde{R} have a rather special relationship: the polar plane of a point $p \in R$ is a tritangent plane to \tilde{R} and vice-versa. Furthermore the polar line of the tangent line through p is a trisecant of \tilde{R} . This classical result can be found in [14]. It establishes a $3 : 3$ correspondence between the two curves.

8 The trichotomy for \mathbf{P}^3

So far we have introduced three pieces of geometry – the vector bundle on a harmonic cubic curve, the Mukai-Umemura threefold, and the projective space \mathbf{P}^3 of the permutation representation of A_5 . We seek the distinguished $SO(3, \mathbf{C})$ -invariant subvarieties in the six-dimensional projective space $\mathbf{P}(V)$ which describe the three different types of bundles. To do this we first restrict to $\mathbf{P}^3 \subset \mathbf{P}(V)$, where we shall see the classical geometry described in the previous section playing a role. The first step is to establish a rational correspondence between the Mukai-Umemura threefold and \mathbf{P}^3 .

8.1 From Mukai-Umemura to \mathbf{P}^3

The Mukai-Umemura threefold parametrizes three-dimensional isotropic subspaces of V . Fix a nondegenerate one, U_0 , defined by the icosahedral set $[a_1], [a_2], \dots, [a_6]$ as a base point. For any other U we can consider the subspace $U + U_0 \subset V$. Generically this has dimension 6 and so has a one-dimensional orthogonal complement spanned by a harmonic cubic f : f then has two orthogonal isotropic subspaces, U and U_0 .

This establishes a rational map from Z to $\mathbf{P}^3 = \mathbf{P}(U_0^\perp)$, but we must examine the indeterminacy.

Proposition 3 *If $\dim(U + U_0) < 6$ then it is defined by an icosahedral set with a point in common with $[a_1], [a_2], \dots, [a_6]$.*

Proof: Suppose that $\dim(U + U_0) < 6$, then $\dim(U + U_0)^\perp \geq 2$ and there is a pencil of cubics passing through the two icosahedral sets. Suppose that the icosahedral set defined by U is nondegenerate: $[b_1], [b_2], \dots, [b_6]$. If two axes of an icosahedron coincide then so do all of the axes, so if the icosahedra are distinct there must be at least 11 points for the pencil to pass through, but by Bézout's theorem, unless there is a common component in the pencil, the maximum number of intersections is nine : we deduce that there must be a common component.

If it is a line, then since no three of the points $[a_i]$ or $[b_i]$ are collinear there is a maximum of four – two from each set – on the line. But then we have seven remaining for the pencil of conics. By Bézout again we reach a contradiction. If the common component is a conic, then it contains at most five points from each set, and the remaining point must lie in a pencil of lines and so for some i, j , $[a_i] = [b_j]$. The equation is therefore of the form

$$(c, x)((a, x)^2 - \frac{1}{5}(a, a)(x, x)) \quad (3)$$

where $[a] = [a_i] = [b_j]$ and $(c, a) = 0$.

If f is orthogonal to a degenerate isotropic subspace then the cubic is tangential to the conic Q at a point $[a]$ and meets the tangent line at a point $[b]$. The pencil thus consists of the pencil of planes in \mathbf{P}^3 containing the tangent line to the rational curve $R \cong Q$ at the corresponding point p . Unless the tangent line is contained in the Clebsch cubic S , the generic curve in the pencil is smooth at p , and we can apply the Cayley-Bacharach theorem (in the degenerate case as in [9] p.672) to deduce that any cubic which passes through the six points $[b], [a_2], [a_3], \dots, [a_6]$ and is also tangent to Q at $[a]$ must pass through $[a_1]$. But take the cubic which is the conic through $[a_2], [a_3], \dots, [a_6]$ together with the tangent at $[a]$:

$$(a, x)((a_1, x)^2 - \frac{1}{5}(a_1, a_1)(x, x)) = 0.$$

If a_1 lies on this then $(a, a_1) = 0$. Repeating for the other $[a_i]$ we deduce that they are all collinear, which is a contradiction.

The tangents to R which are contained in S are the lines \tilde{E}_i , and the pencil is the same as (3). \square

For each vertex a_i of an icosahedron, Proposition 3 has highlighted the role of the icosahedra with this as a common vertex. They form an orbit of the group of rotations fixing a_i – a \mathbf{C}^* orbit in Z . Its closure is a rational curve with two extra points, fixed by the action, which lie in the divisor D (they are two type 1 degenerate icosahedra, determined by the two tangent lines to Q from $[b] = [a_i]$). These six rational curves C_i all pass through the basepoint U_0 in Z . From Proposition 3 there is a well-defined map from the complement of these six curves to \mathbf{P}^3 :

$$\beta(U) = (U + U^0)^\perp.$$

We shall extend this to Z by blowing up certain subvarieties.

- (i) First we need to deal with the basepoint $U = U_0$. Blow up the basepoint in Z , replacing it by the projectivized tangent space. Since the stabilizer of U_0 is finite, the tangent space is naturally $\mathfrak{so}(3)$. For each $b \in \mathfrak{so}(3) \cong \mathbf{C}^3$ we orthogonally project f_b onto U_0^\perp to get p_b . If this projection is zero then $(f, f_b) = f(b) = 0$ for all $f \in U_0$, so that any cubic which vanishes at the a_i also vanishes at b . However, these cubics define a projective embedding of the blown-up plane, and so must separate points. It follows that the projection is zero if and only if $[b] = [a_i]$ for some $1 \leq i \leq 6$. We can therefore extend β to $\mathbf{P}^2 \setminus \{[a_1], [a_2], \dots, [a_6]\}$ by using the orthogonal projection p_b .
- (ii) The six proper transforms of the curves C_i meet the exceptional divisor in the points $[a_i]$ and we need to blow up these curves to extend the map. Consider $p_{b(t)}$ for $b(t) = a_i + tv$ as $t \rightarrow 0$, where $(v, a_i) = 0$. We have

$$f_{b(t)} = f_{a_i} + 3t(v, x) \left((a_i, x)^2 - \frac{1}{5}(a_i, a_i)(x, x) \right) + \dots$$

Now f_{a_i} lies in U_0 , so

$$p_{b(t)} = 3t(v, x) \left((a_i, x)^2 - \frac{1}{5}(a_i, a_i)(x, x) \right) + \dots$$

and the coefficient of t extends β to the blow-up of the projectivized tangent space at $[a_i]$. Its image in \mathbf{P}^3 is the Clebsch cubic. The exceptional curve E_i obtained by blowing up $[a_i]$ maps to the pencil in Proposition 3.

Denote by \hat{Z} the blown-up Mukai-Umemura threefold.

Remark: Blowing up C_i picks out a cubic of the form

$$(c, x)((a_i, x)^2 - \frac{1}{5}(a_i, a_i)(x, x)) \tag{4}$$

with $(c, a_i) = 0$ which extends the map β . The normal bundle of C_i is trivial so the blow-up is $C_i \times \mathbf{P}^1 \cong \mathbf{P}^1 \times \mathbf{P}^1$ and β collapses the first factor. This corresponds to the fact that the cubic (4) contains the one-parameter family of icosahedral sets which include $[a_i]$. In general the positive-dimensional fibres of the map β give us cubics containing infinitely many icosahedral sets.

8.2 The trisecant surface

The image $\beta(D) \subset \hat{Z}$ in \mathbf{P}^3 consists of those polynomials $f \in U_0^\perp$ which are also orthogonal to a degenerate isotropic subspace. From Proposition 2 this is given by the vanishing of an A_5 -invariant polynomial of degree 10, the restriction of the $SO(3, \mathbf{C})$ -invariant polynomial Δ .

On the other hand, we know that if f is orthogonal to a degenerate isotropic subspace, the cubic curve C is defined by a plane section of S which is tangent to the rational curve R . This means that $[f] \in \mathbf{P}^3$ lies on the polar line of a tangent line to R . From the special properties of the curves R and \tilde{R} observed in Section 7 this is a *trisecant* of \tilde{R} . So the degree 10 polynomial vanishes on the trisecant surface T of \tilde{R} .

Now a generic line meets d trisecants of \tilde{R} if its polar line ℓ meets d tangents to R . But this surface is the tangent developable surface and we can calculate the degree easily. If X is a curve of degree n and $\mathbf{C}^2 \subset H^0(\mathbf{P}^3, \mathcal{O}(1))$ is the pencil of planes through a line ℓ then under the one-jet evaluation map

$$ev : H^0(\mathbf{P}^3, \mathcal{O}(1)) \rightarrow J^1(X, \mathcal{O}_{\mathbf{P}^3}(1))$$

we have a section of

$$\Lambda^2(J^1(X, \mathcal{O}_{\mathbf{P}^3}(1))) \cong K_X(2)$$

and the degree of this is $2g - 2 + 2n$. It vanishes at the points of intersection of ℓ and the surface. In our case R is of degree 6 and rational so the developable has degree $0 - 2 + 12 = 10$.

Thus the trisecant surface, which is irreducible, is of degree 10 and given by the vanishing of the invariant polynomial Δ of Proposition 2. It is the locus of cubics which pass through a nondegenerate icosahedral set $[a_1], [a_2], \dots, [a_6]$ and a degenerate one.

8.3 The three cases for \mathbf{P}^3

Consider now the trichotomy in the light of the construction of the map $\beta : \hat{Z} \rightarrow \mathbf{P}^3$. The trisecant surface $T = \beta(D)$ consists of cubics which contain a nondegenerate

icosahedral set and a degenerate one. They contain therefore two or infinitely many. The image of the six blown-up curves C_i have infinitely many. This image moreover lies in the Clebsch cubic. There only remains the rest of the Clebsch cubic, for which there is either one icosahedral set or infinitely many. The whole surface is therefore the locus of sections of \mathcal{E}^* in **4** which have a degenerate zero set – the second and third cases of the trichotomy.

We wish finally to determine which cubic curves contain infinitely many icosahedral sets, that is, which of the sections of \mathcal{E}^* on Z vanish on a positive-dimensional variety $Y \subset Z$. Since $c_1(Z) > 0$, Y must intersect the anticanonical divisor D nontrivially, and so we are looking for $[f]$ lying in S and also in the trisecant surface T .

Now the special property of \tilde{R} we observed is that the polar of any point on it is a tritangent plane of R . This means that $[f] \in \tilde{R}$ defines a plane section of $S \subset \mathbf{P}^3$ which meets the curve R tangentially at three points. In other words the associated plane cubic C passes through three and hence infinitely many icosahedral sets. A trisecant line of \tilde{R} meets the cubic surface S generically in three points, which are the three points of intersection with \tilde{R} . Thus for these lines T intersects S only in \tilde{R} .

There are degenerate cases – twelve points (an A_5 -invariant set) on \tilde{R} where the tritangent plane to R meets it in two points with multiplicity three instead of three with multiplicity two. The trisecant line to \tilde{R} at these points lies entirely in the Clebsch cubic – these are the six exceptional divisors E_1, E_2, \dots, E_6 .

Set-theoretically, T intersects S in \tilde{R} and the six lines. We have therefore shown:

Theorem 4 *A generic point of \mathbf{P}^3 defines a cubic which contains precisely two icosahedral sets. If it contains one or infinitely many it lies on the Clebsch cubic surface S and in the latter case it lies on the degree six rational curve $\tilde{R} \subset S$ or on the six lines E_1, \dots, E_6 .*

Remark: In terms of divisor classes let H be the pull-back of the hyperplane divisor on the original projective plane then a plane cubic C through $[a_1], \dots, [a_6]$ lifts to a curve whose divisor is $3H - (E_1 + \dots + E_6)$. The embedding for a cubic surface in \mathbf{P}^3 is the anticanonical embedding so

$$-K_S \sim 3H - (E_1 + \dots + E_6).$$

The divisor of R , the lift of a conic in \mathbf{P}^2 , is $2H$.

Now consider the other six exceptional curves coming from the conics passing through five of the six points. These have classes $\tilde{E}_1 \sim 2H - (E_2 + \dots + E_6)$ etc. and

$$-K_S \sim 3\tilde{H} - (\tilde{E}_1 + \dots + \tilde{E}_6) = 3H - (E_1 + \dots + E_6)$$

It follows that $\tilde{H} \sim 5H - 2(E_1 + \cdots E_6)$, where \tilde{H} is the hyperplane divisor for $\tilde{\mathbf{P}}^2$. Then $\tilde{R} \sim 2\tilde{H} \sim 10H - 4(E_1 + \cdots E_6)$, so the divisor class of T on S is

$$-10K_S \sim 10(3H - (E_1 + \cdots + E_6)) = 3(10H - 4(E_1 + \cdots E_6)) + 2(E_1 + \cdots E_6).$$

We may also note here that

$$R + \tilde{R} \sim 2H + 10H - 4(E_1 + \cdots E_6) = 4(3H - (E_1 + \cdots + E_6)) \sim -4K_S$$

so that there is an invariant quartic surface which vanishes on the pair of curves. This is $9\sigma_2^2 - 20\sigma_4$.

9 The trichotomy for $\mathbf{P}(V)$

Since a generic $f \in V$ defines a cubic which contains a finite number of icosahedral sets, and since the stabilizer of an icosahedron is finite, then $\mathbf{P}^3 \subset \mathbf{P}(V)$ sweeps out an open set under the action of $SO(3, \mathbf{C})$. In principle, therefore, the trichotomy for $\mathbf{P}(V)$ entails looking for a hypersurface which intersects \mathbf{P}^3 in the Clebsch cubic. However, the Hilbert polynomial for the four-dimensional representation space of A_5 is

$$\frac{1 + t^{10}}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^5)}$$

and for the $SO(3, \mathbf{C})$ invariants in the seven-dimensional representation space

$$\frac{1 + t^{15}}{(1 - t^2)(1 - t^4)(1 - t^6)(1 - t^{10})}$$

so there is no degree 3 invariant for $SO(3, \mathbf{C})$. However, as shown in [12], there is an invariant sextic hypersurface in $\mathbf{P}(V)$ which meets \mathbf{P}^3 tangentially in the Clebsch cubic. The formula given in [12] is adapted to the context of spherical harmonics and functions on the two-sphere, but here we shall adopt the point of view in [13] instead, using the realization of the representation space V as S^6 – homogeneous sextic polynomials in (z_1, z_2) .

In inhomogeneous form the sextic is written as

$$u_0x^6 + u_1x^5 + \cdots + u_6$$

with roots x_1, x_2, \dots, x_6 . Writing (ij) for $x_i - x_j$ the discriminant is the degree 10 invariant

$$\Delta = u_0^{10} \prod_{i < j} (ij)^2.$$

As we have seen, the vanishing of this describes the locus of cubics f which are orthogonal to a degenerate isotropic subspace. In [13], Igusa defines similar invariants, summing over permutations:

$$\begin{aligned} A &= u_0^2 \sum_{\text{fifteen}} (12)^2 (34)^2 (56)^2 \\ B &= u_0^4 \sum_{\text{ten}} (12)^2 (23)^2 (31)^2 (45)^2 (56)^2 (64)^2 \\ C &= u_0^6 \sum_{\text{sixty}} (12)^2 (23)^2 (31)^2 (45)^2 (56)^2 (64)^2 (14)^2 (25)^2 (36)^2 \end{aligned}$$

and introduces rational invariants $J_2, J_4, J_6, J_8, J_{10}$ where

$$J_6 = \frac{A^3}{221184} + \frac{5AB}{13824} - \frac{C}{576}$$

and $J_{10} = 2^{-12} \Delta$.

Theorem 5 *A generic point of $\mathbf{P}(V)$ defines a cubic which contains precisely two icosahedral sets. It contains one or infinitely many if and only if it lies on the hypersurface $J_6 = 0$.*

Proof: The proof consists of using the explicit forms in Section 4 of [13] to determine a degree six invariant which vanishes on the $SO(3, \mathbf{C})$ transforms of the rational curves \tilde{R} and E_1, \dots, E_6 . This can be done by evaluating on the normal forms below:

- (i) A point in \tilde{R} gives a sextic polynomial in z with three double zeros, and each such is equivalent under the action of $SO(3, \mathbf{C})$ to $z^2(z-1)^2$.
- (ii) The cubic polynomials in E_i are of the form

$$(c, x)((a_i, x)^2 - \frac{1}{5}(a_i, a_i)(x, x))$$

with $(c, a) = 0$. The line $(a, x) = 0$ intersects Q given by $(x, x) = 0$ at two points in general and c is a third point on this line. Its polar line $(c, x) = 0$ intersects Q in two more points, so the sextic has two double zeros and two simple ones if $[c]$ does not lie on Q . There is a constraint however – identifying Q with \mathbf{P}^1 the four points must have cross-ratio -1 . A normal form for this is $z^2(z^2 - 1)$.

We finally need to consider the points of $\mathbf{P}(V)$ whose $SO(3, \mathbf{C})$ orbits do not intersect \mathbf{P}^3 . This means cubics which do not contain any nondegenerate icosahedral set. If

the cubic contains two degenerate icosahedral sets, then it is tangential twice to the null conic, so in the sextic polynomial interpretation we can take it to the form

$$z^2(az^2 + bz + c).$$

and evaluating J_6 gives

$$\frac{1}{1024}b^2(b^2 - 4ac)^2.$$

If $J_6 = 0$, then either $b^2 - 4ac = 0$ which is Case (i) above or $b = 0$ which is Case (ii). \square

10 A special cubic curve

We remarked above that the sextic polynomial $z^2(z - 1)^2$ is a normal form for an element in S^6 which represents a cubic curve containing an infinite number of icosahedral sets. There is thus a single $SO(3, \mathbf{C})$ orbit for a cubic which is tangential to the null conic at three distinct points. Perhaps the simplest canonical form for this is

$$y^2 = x^3 + x^2 + 4x + 4$$

where the null conic has equation

$$y^2 = 4(2x - 3)(x + 1).$$

The two meet at just three points: $(4, 10), (4, -10), (-1, 0)$.

Note that this cubic C is defined over the integers and has rational points

$$\infty, (4, -10), (0, -2), (-1, 0), (0, 2), (4, 10)$$

forming a cyclic group of order six.

Now C contains a one-parameter family of icosahedral sets, three of which are degenerate. Moreover this family is rational. To see this, we revert to the isotropic subspace description. From Proposition 3 we may assume that there are two isotropic subspaces U_1, U_2 of W for which $U_1 \cap U_2 = 0$. A third subspace U_3 intersects each of these trivially and hence is the graph of a linear transformation $S : U_1 \rightarrow U_2$. Thus for $w_1, w_2 \in U_1$ we have

$$0 = \omega_x(w_1 + Sw_1, w_2 + Sw_2) = \omega_x(w_1, Sw_2) + \omega_x(Sw_1, w_2)$$

since $\omega_u(w_1, w_2) = 0 = \omega_u(Sw_1, Sw_2)$. This means the graph of tS for any $t \in \mathbf{C}$ is also isotropic, so we have a family of isotropic subspaces with rational parameter t .

Each icosahedral set consists of six points and so we have a sixfold covering $p : C \rightarrow \mathbf{P}^1$ with three branch points $0, 1, \infty$ in \mathbf{P}^1 . At these points the icosahedron degenerates – one axis remains (the vector b in the type 1 degenerate subspace) and the other five coalesce to be a null vector (the vector a). The inverse image of each branch point consist of two points – one, $[a]$, a ramification point of order 5 and one simple point $[b]$. These six points are precisely the six rational points in the canonical form above.

The projection map p is given in its simplest form by the meromorphic function

$$p^2 = \frac{(1+x)^5(y-x-2)}{(y+3x-2)^5}.$$

An explicit form for p is a little less simple – it involves elliptic functions for two planar embeddings (see Remark 5 in Section 12), which differ by the line bundle L of order 2, for here we are considering a curve where the vector bundle is $L \oplus L$ with L^2 trivial.

Remark: The formulae above have come from [17] in the context of algebraic solutions to the hypergeometric equation. The monodromy of the covering p lies in the icosahedral group and mapping into $PSL(2, \mathbf{C})$ this gives a second order differential equation with three singular points. Schwarz’s celebrated paper [16] classifying these stops short of explicit expressions and the elliptic curve above appears in an attempt to be more concrete about these.

11 Higher degree curves

The picture for higher degree representations and curves is not clear – in particular there are stable bundles and not just semi-stable as in the case of the elliptic curve. One might ask for a link between semistability and isotropic subspaces, however. The argument in Section 3 generalizes to show that a d -dimensional isotropic subspace orthogonal to a harmonic polynomial f of degree d defines a line subbundle L with $\deg L = \deg E/2$, though it is not clear if the reverse is true. What is true is that isotropic subspaces are rare. We can find some, though, as we show below.

Let V now be the $(2d+1)$ -dimensional irreducible representation space of $SO(3, \mathbf{C})$ and consider a d -dimensional subspace isotropic for all skew forms $(x \cdot u, v)$ which is invariant by $SO(2, \mathbf{C}) \subset SO(3, \mathbf{C})$. Then it is a sum of d weight spaces L_m where $-d \leq m \leq d$. Let $\mathbf{h}, \mathbf{n}^+, \mathbf{n}^-$ be a basis of $\mathfrak{so}(3)$ where \mathbf{h} generates $SO(2, \mathbf{C})$ and $[\mathbf{h}, \mathbf{n}^+] = \mathbf{n}^+, [\mathbf{h}, \mathbf{n}^-] = -\mathbf{n}^-$.

Proposition 6 *Let $U \subset V$ be a d -dimensional $SO(2, \mathbf{C})$ -invariant isotropic subspace of V . Then U is a sum of weight spaces $\pm\{1, 2, 3, \dots, d\}$ or $\pm\{0, 2, \dots, d\}$*

Proof: Because \mathbf{h} acts the scalar m on L_m , if a weight $m > 0$ occurs, then $-m$ does not because on $L_m \oplus L_{-m}$, $(\mathbf{h} \cdot u, v) = muv$ which is non-zero.

Now $\mathbf{n}^+ : L_m \mapsto L_{m+1}$, so on $L_m \oplus L_{-m-1}$, $(\mathbf{n}^+ \cdot u, v) = uv$ which is non-zero. Hence if m occurs, $-m - 1$ does not. Similarly with \mathbf{n}^- , if m occurs $-m + 1$ does not.

Suppose 0 is not one of the d weights and let P be the positive weights. Then the negative weights N must be the negatives of the complement of P in $\{1, \dots, d\}$ by the first criterion. If $m \in P$ then by the second criterion, the adjacent numbers $m - 1, m + 1$ must be in P . But this means that P consists of $1, 2, 3, \dots, d$. If P is empty, then we get $-1, -2, -3, \dots, -d$

If 0 is one of the weights, then by the second criterion ± 1 is not a weight. Then P is a subset of $2, \dots, d$. Again the $d - 1$ non-zero weights must be made up of P and the negative of its complement. As before adjacent positive numbers must be in P , so P is $2, 3, \dots, d$ so the weights are $0, 2, \dots, d$ or their negatives. \square

Note that these are precisely the *degenerate* isotropic subspaces which we have encountered for $d = 3$. So if the curve $f(x) = 0$ has a high order contact with the conic Q , we may deduce that the bundle E is strictly semistable.

12 Further remarks

1. We have shown here that a generic cubic which passes through one icosahedral set passes through another one. According to Melliez [14] (who also introduced the use of the bundle \mathcal{E}^*), this is a result of Reye and can be found in Baker [2], page 145, Exercise 26 or Coble [6] page 236. The reader is invited to make the translation.
2. A discussion of the real case can be found in [12]. There we consider the picture in the sphere double covering \mathbf{RP}^2 . The inverse image of the cubic curve is then the so-called *nodal set* of a spherical harmonic and the issue is whether the nodal set contains the vertices of a regular icosahedron. The rational curves R and \tilde{R} are not real and so do not appear in the story.
3. The curve \tilde{R} is simply the null conic in $\tilde{\mathbf{P}}^2$ but is much more complicated when viewed in \mathbf{P}^2 . We saw from Remark 8.3 that its divisor class is $10H - 4(E_1 + \dots + E_6)$ so It is of degree 10. In fact it is the invariant singular curve defined by: $2(x_1^{10} + x_3^{10} + x_2^{10}) + 35(x_1^8 x_2^2 + x_1^2 x_2^8 + x_2^2 x_3^8 + x_2^8 x_3^2 + x_3^8 x_1^2 + x_3^2 x_1^8) + 25\sqrt{5}(x_1^2 x_2^8 + x_2^2 x_3^8 +$

$$x_3^2x_1^8 - x_1^8x_2^2 - x_2^8x_3^2 - x_3^8x_1^2) - 30(x_1^6x_2^4 + x_1^4x_2^6 + x_2^6x_3^4 + x_2^4x_3^6 + x_3^6x_1^4 + x_3^4x_1^6) + 50\sqrt{5}(x_1^6x_2^4 + x_2^6x_3^4 + x_3^6x_1^4 - x_1^4x_2^6 - x_2^4x_3^6 - x_3^4x_1^6) - 560x_1^2x_2^2x_3^2(x_1^4 + x_2^4 + x_3^4) + 1060x_1^2x_2^2x_3^2(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2) = 0.$$

4. The closure of the curves in the Mukai-Umemura threefold defined by icosahedra with a fixed axis are rational curves C with $c_1(Z)[C] = 2$, i.e degree 2. These are the minimal rational curves appearing in the classification of Fano varieties. The one-parameter family of icosahedra parametrized by points of \tilde{R} define degree 3 curves, for the plane cubic C is tangential to the null conic at three points. Whereas degree 3 rational curves were linked to icosahedral solutions to the hypergeometric equation, degree 4 curves give rise to algebraic solutions of a particular case of Painlevé's sixth equation. This is the setting of [10],[11] where the dihedral and octahedral cases are examined. A classification of all such solutions, akin to Schwarz's list, has been given by Boalch [4].

To find degree 4 rational curves we consider the map $\beta : \hat{Z} \rightarrow \mathbf{P}^3$ of Section 8.1. Since \tilde{R} parametrizes cubics containing an infinite number of icosahedral sets, the map β collapses a subvariety in \hat{Z} . If we want to go the other way, from \mathbf{P}^3 to Z , we first blow up \tilde{R} and then do some blowing up and down. Now take a *secant* to the curve $\tilde{R} \subset \mathbf{P}^3$ – a line joining two points. A generic line intersects the trisecant surface T in 10 points. However, the 3 : 3 correspondence between R and \tilde{R} implies that through a generic point of \tilde{R} there pass three trisecants. Blowing up \tilde{R} means that the proper transform of the secant is a curve which meets the blown-up trisecant surface in $10 - 3 - 3 = 4$ points, and this becomes a rational curve of degree 4 in the Mukai-Umemura threefold. The corresponding solution to Painlevé VI is known and is due to Dubrovin and Mazzocco [8]. An open set in the Mukai-Umemura threefold can then be identified with the twistor space of a complex self-dual four-manifold and the secants above, after transforming by $SO(3, \mathbf{C})$, give the four-parameter family of twistor lines.

5. One might ask, without reference to the vector bundle E , how a cubic curve defined by a harmonic polynomial gives rise to a line bundle L . It arises from the geometry as follows.

As a varies in the cubic C consider the polynomial f_a , invariant under the action of $a \in \mathfrak{so}(3)$. Now $\omega_a(f_a, v) = (a \cdot f_a, v) = 0$ so that f_a generates a line bundle in the kernel of $\omega : W \rightarrow W^*(1)$. We identify this bundle as follows. Consider the map $\gamma : \mathbf{P}^2 \rightarrow \mathbf{P}(V)$ defined by $a \mapsto f_a$. Then f_a spans the pull-back $\gamma^*(\mathcal{O}(-1))$ and since f_a is homogeneous of degree 3 in a , $\gamma^*(\mathcal{O}(-1)) \cong \mathcal{O}_{\mathbf{P}^2}(-3)$. Restrict this to C to get the required line bundle.

We then have an inclusion $\mathcal{O}(-3) \subset E^*(-1)$ over C . Projecting to $E^*(-1)/L(-1)$

gives a section of $L^*(2)$. This vanishes when $f_a \in U$ which is at the six points $[a_1], \dots, [a_6]$ defined by the icosahedral set. Thus on C we have the relation of divisor classes

$$L \sim \mathcal{O}(2) - \sum_1^6 [a_i].$$

Now we saw that the cubics through the icosahedral set are given by plane sections of the Clebsch cubic surface. This means that the same cubic is embedded in two different planes – a plane in \mathbf{P}^3 and the original plane with the conic Q . Since the embedding in \mathbf{P}^3 is

$$\mathcal{O}(3) - \sum_1^6 [a_i]$$

we see that the degree zero line bundle L is the *difference* of the two hyperplane divisor classes.

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